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# ON AN INVERSE PROBLEM RELATIVELY POTENTIAL AND ITS SOLUTION

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**Abstract.** In this work the wave equation is analytically solved in the variational form and for the gradient of the functional the analytical expression is found. Also analytical expression for optimal potential in inverse potential is obtained.

**Keywords:** Inverse problem, variational method, gradient, optimal potential.

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## 1. Problem statement

It is known that the motion of a particle in a central field is described by the equation

$$-\frac{a}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{bR}{r^2}+q(r)R=ER\ . \tag{1}$$

Here $a>0$ and $b$ are given numbers, $q(r)$ is the energy of interaction. Multiplying this equation by the $r^2$ and denoting by

$$Q(r)=b+q(r)r^2,$$

we obtain

$$-a\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+Q(r)R=Er^2R\,. \tag{2}$$

One of the interesting problems is to obtain an analytical solution of for different and special potentials $Q(r)$ that is not always possible. Also one of the main problems is the solution of inverse problem, finding of the potential $Q(r)$ on the given energy eigenvalues of the Eq.(2), if possible:

$$R(r_0)=z_0,\ R(r_1)=z_1,\ R(r_2)=z_2,\dots,R(r_n)=z_n \tag{3}$$

where $0<r_0<r_1<\dots<r_n;\quad n\ge 2$.

Now we consider the equation (2) on the interval $[r_1, r_n]$. Our aim is to find here the potential $Q(r)$ in the interval $[r_1, r_n]$, such that the solution of the problem (2) $R(r)$ satisfies the condition (3).

**Variational formulation of the problem.** Here we assume that the solution of the problem posed in (2), (3) exists. In order to solve the problems (2), (3), we write it in the variational form: Find the minimum of the functional

$$J(Q) = \sum_{i=1}^{n-1}[R(r_i) - z_i]^2 \to \min, \tag{4}$$

under condition (2) and

$$R(r_0) = z_0, \;\; R(r_n) = z_n. \tag{5}$$

It is clear, that if $Q^* = Q^*(r)$ is a solution of the problem (2), (3), it is also a solution of the variational problem (2), (4), (5). The converse statement is also true. (The converse is also inverse).

Indeed, let $Q^* = Q^*(r)$ be a solution of (2), (4), (5). Then $J(Q^*) = 0$, because of the conditions (3). The case $J(Q^*) > 0$ contradicts to the condition of the problem (2), (3).

Considering the equivalence of the problems (2), (3) and (2), (4), (5), here we will investigate the optimal control problem (2), (4) and (5). Management role in this task plays by the potential $Q = Q(r),\ r \in [r_0, r_n]$. Usually on the $Q(r)$ additional conditions are set [4].

**Finding of the optimal potential.** Let the class of controls $Q(r)$ be in the form:

$$U = \{Q = Q(r) \in L_2(r_0, r_n) : Q_n \le Q(r) \le Q_1,\ \forall r \in [r_0, r_n]\}. \tag{6}$$

Here $0 \le Q(r) \le Q$ are given numbers.

Let $\psi = \psi(r)$ be a solution of the problem

$$-a\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + Q(r)\psi - Er^2\psi = -2\sum_{i=1}^{n-1}[R(r) - z_i]\delta(r - r_i), \tag{7}$$

$$\psi(r_0) = 0, \quad \psi(r_n) = 0. \tag{8}$$

Here $\delta = \delta(t)$ - is the Dirac delta function with properties

$$\delta(t) = \begin{cases} +\infty, & t = 0, \\ 0, & t \ne 0. \end{cases}$$

Suppose $Q = Q(r),\ \overline{Q} = \overline{Q}(r)$ that two any of potential of the plurality $U$. Solution of the problems (2), (5) corresponding to these potentials, we will denote by $R(r)$ and $\overline{R}(r)$, respectively. Then it is clear that

$$-a\frac{d}{dr}\left(r^2\frac{d\Delta R}{dr}\right) + Q(r)\Delta R - Er^2\Delta R + \Delta Q\overline{R} = 0, \tag{9}$$

$$\Delta R(r_0)=0,\ \ \Delta R(r_n)=0, \tag{10}$$

where

$$\Delta R=\overline{R}(r)-R(r),\ \ \Delta Q=\overline{Q}(r)-Q(r).$$

Multiplying (9) by $\psi=\psi(r)$ and integrating it over the interval $[r_0,r_n]$ we get

$$\begin{aligned}&\int_{r_0}^{r_n}\left[-a\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)+Q(r)\psi-Er^2\psi\right]\Delta R dr+\int_{r_0}^{r_n}\Delta Q\overline{R}\,\psi dr-\\&-ar^2\frac{d\Delta R}{dr}\psi\Big|_{r_0}^{r_n}+ar^2\frac{d\psi}{dr}\Delta R\Big|_{r_0}^{r_n}=0.\end{aligned} \tag{11}$$

Now we calculate the increment of the functional (4)

$$\begin{aligned}\Delta J&=J(\overline{Q})-J(Q)=\sum_{i=1}^{n-1}\left[\overline{R}(r_i)-z_i\right]^2-\sum_{i=1}^{n-1}\left[R(r_i)-z_i\right]^2=\\&=2\sum_{i=1}^{n-1}\left[R(r_i)-z_i\right]\Delta R(r_i)+\sum_{i=1}^{n-1}\left[\overline{R}(r_i)-R(r_i)\right]^2.\end{aligned} \tag{12}$$

Adding to (12) equation (11) (since the right-hand side is equal to zero), and given that $\Delta R(r_0)=\Delta R(r_n)=0$ we have

$$\begin{aligned}\Delta J&=2\sum_{i=1}^{n-1}\left[R(r_i)-z_i\right]\Delta R(r_i)+\int_{r_o}^{r_n}\Delta Q\overline{R}\,\psi\,dr+\\&+\int_{r_0}^{r_n}\left[-a\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)+Q(r)\psi-Er^2\psi\right]\Delta R dr-ar^2\frac{d\Delta R}{dr}\psi\Big|_{r_0}^{r_n}+\\&+\sum_{i=1}^{n-1}\left[\overline{R}(r_i)-R(r_i)\right]^2=0.\end{aligned} \tag{13}$$

Here we use the known properties of the $\delta(t)$ function of the Dirac

$$\int_{r_0}^{r_n}f(r)\delta(r-r_i)dr=f(r_i),\ i=\overline{1,n-1}.$$

Then we can write

$$\sum_{i=1}^{n-1}\left[R(r_i)-z_i\right]\Delta R(r_i)=\sum_{i=1}^{n-1}\int_0^T[R(t-z_i]\Delta R(t)\delta(t-r_i)dt. \tag{14}$$

Taking this into account in (13) and the fact that $\psi=\psi(r)$ is the solution of the problems (7), (8), (13) we obtain

$$\Delta J=\int_{r_0}^{r_n}Q(r)\psi\,\Delta R dr+\sum_{i=1}^{n-1}\left[\overline{R}(r_i)-R(r_i)\right]^2. \tag{15}$$

Now we can show that

$$|\Delta R(r)|\le c\|\Delta Q\|_{L_2(r_0,r_n)}, \tag{16}$$

where $c > 0$ is constant.

For this purpose we multiply the equation (9) by $\Delta R$ and integrate it on $[r_0, s],\ r_0 < s \le r_n$ .

Then we obtain

$$a\int_{r_0}^{s} r^2 \left(\frac{d\Delta R}{dr}\right)^2 dr + \int_{r_0}^{s} [Q(r) - Er^2]\Delta R^2 dr + \int_{r_0}^{s} \Delta Q \overline{R} \Delta R dr = 0 .$$

From this expression it is clear that

$$ar_0^2 \int_{r_0}^{s} \left(\frac{d\Delta R}{dr}\right)^2 dr \le C_1 \|\Delta Q\|^2 + C_2 \int_{r_0}^{s} |\Delta R|^2 dr .$$

Here $C_1, C_2$ are positive constants. Then,

$$\int_{r_0}^{s} \left(\frac{d\Delta R}{dr}\right) \le C_3 \|\Delta Q\|^2 + C_4 \int_{r_0}^{s} |\Delta R|^2 dr. \tag{17}$$

Taking into account that $\Delta R(r_0) = 0$ it is possible to write

$$(\Delta R(s))^2 = \int_{r_0}^{s} \frac{d}{dr}(\Delta R)^2 dr = 2\int_{r_0}^{s} \Delta R \frac{d\Delta R}{dr} dr \le$$
$$\le \int_{r_0}^{s} (\Delta R)^2 dr + \int_{r_0}^{s} \left(\frac{d\Delta R}{dr}\right)^2 dr.$$

Here we take into account (17), and find

$$(\Delta R(s))^2 \le C_6 \int_{r_0}^{s} (\Delta R)^2 dr + C_7 \|\Delta Q\|^2 . \tag{18}$$

Using Gronwall's lemma [6] from (18) we get

$$(R(s))^2 \le C_8 \|\Delta Q\|^2, \quad s \in [r_0, r_n].$$

We have proved the inequality Taking this fact into account in we have

$$\Delta J = \int_{r_0}^{r_n} Q \psi \Delta R dr + o(\|\Delta Q\|). . \tag{19}$$

Using standard techniques of Ref.[6] from (19) we obtain the following theorem.

**Theorem 1.** Let $Q^* = Q^*(r)$ be the optimal potential for the problem (2), (4), (5). Then for any $Q = Q(r) \in U$ the relations

$$Q^*(r)\,\psi^*(r)R^*(r) = \min Q(r)\psi^*(r)R^*(r), \ \ \forall r \in [r_0, r_n],\ Q_0 \le Q(r) \le Q_1 , \tag{20}$$

are true. Here $R^* = R^*(r)$, $\psi^* = \psi^*(r)$, solutions of a problem (2), (5) and (7), (8) at $Q = Q(r)$.

**Theorem 2.** Functional (4) is differentiable and its gradient is given by the formula

$$J'(Q) = \psi R .$$

Theorem 1 allows us to determine the optimal potential analytically.

**Consequence.** Let $Q^* = Q^*(r)$ be the optimal potential for the problem (2), (4), (5). Then

$$Q^*(r) = \begin{cases} Q_0, & \psi^*(r) > 0, \\ Q_1, & \psi(r) < 0. \end{cases}$$

In the case $\psi^*(r) = 0$, the potential can be chosen arbitrarily.

## References


1. Greiner W. Quantum Mechanics, 4th edition, Berlin, Springer, 2001, 512 p.
2. Bagrov V.G., Gitman D.M., Exact Solutions of Relativistic Wave Equations, Kluwer Academic Publishers, Dordrecht, 1990, 323p.
3. Akhmedov Kh.I., Guseinova N.Sh., An analytical solution of the Klein-Fock-Gordon equation for a 2D pion atom moving in a constant uniform magnetic field, Russian Physics Journal, Vol.52, No.3, 2009.
4. Huseynova N.Sh., Niftiyev A.A., On dependence of energy eigenvalue on the ends of the interval, Transactions of ANAS, Vol.31, No.4, 2001, pp.159-166.
5. Ahmadov H.I., Aydin C., Huseynova N.Sh., Uzun O., Analytical solutions of the Schrodinger equation with the Manning-Rosen potential plus a Ring-Shaped like potential, International Journal of Modern Physics E, Vol.22, No.10, 2013, pp.1350072-1-1350072-16.
6. Vasilev F.P., Methods for Solving Extremal Problems, Nauka, Moscow, 1981, 400 p (in Russian).